Classifying Totally Ramified Galois Extensions of Prime Power Order Over Local Fields

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Introduction

We want to determine the Artin-Schreier equations which define Galois extensions.

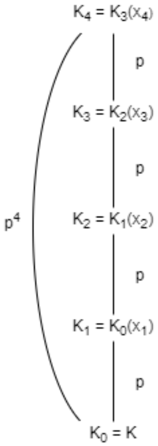
The extensions we are interested in are:

- Over a local field K
- Of characteristic p, an odd prime
- Of degree p^4
- Totally ramified

Definitions

Throughout this presentation, the following definitions will be used:

- $K = K_0$ is the base field.
- K_i is a degree p extension over K_{i-1}.
- $\wp$ is the Weierstrass $\wp$ function, defined as $\wp(x) = x^p - x$.
- x_i is an element of $K_i - K_{i-1}$ such that $\wp(x_i) \in K_{i-1}$.



Presentations of Galois Groups

According to Burnside (2013), there are fifteen Galois groups for extensions of this definition. Five are abelian; ten are nonabelian.

(vi) $\langle a,b,c,d : a^p = b^p = c^p = d^p = 1, cd = dca, bd = db, ad = da, bc = cb, ac = ca, ab = ba\rangle$

(vii) $\langle a,b,c : a^{p^2} = b^p = c^p = 1, ac = ca^{1+p}, ba = ab, bc = cb\rangle$

$(viii)$ $\langle a,b,c : a^{p^2} = b^p = c^p = 1, bc = cba^p, ab = ba, ac = ca\rangle$

(ix) $p = 3$ $\langle a,b,c : a^{p^2} = b^p = c^p = 1, ab = ba, ac = cab, bc = a^{-p}cb\rangle$

$p > 3$ $\langle a,b,c,d : a^p = b^p = c^p = d^p = 1, ac = cab, bc = cbd, cd = dc, ba = ab, ad = da, db = bd\rangle$

(x) $\langle a,b,c : a^{p^2} = b^p = c^p = 1, ac = cab, ab = ba, bc = cb\rangle$

(xi) $\langle a,b,c : a^{p^2} = b^p = c^p = 1, ab = ba^{1+p}, ac = cab, bc = cb\rangle$

(xii) $\langle a,b,c : a^{p^2} = b^p = 1, c^p = a^p, ab = ba^{1+p}, ac = cab, bc = cb\rangle$

$(xiii)$ $\langle a,b,c : a^{p^2} = b^p = 1, c^p = a^{np}, ab = ba^{1+p}, ac = cab, bc = cb, n \text{ is a non-residue } (\text{mod } p)\rangle$

(xiv) $\langle a,b : a^{p^2} = b^{p^2} = 1, ab = ba^{1+p}\rangle$

(xv) $\langle a,b : a^{p^3} = b^p = 1, ab = ba^{1+p^2}\rangle$

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Process Overview

To determine the Artin-Schreier equations for each extension, the following general process was followed:

- Start with equations for degree p^3 extension ${}^{K_3}/_{K}$
  - These were determined by Elder last year
- Observe group structure of $Gal\left({}^{K_4}/_{K_1}\right)$ (degree p^3)
  - This determines all group actions except one
- Determine how σ acts on x_4, where $Gal\left({}^{K_1}/_{K}\right) = \langle\sigma\rangle$
- Use all of this information to determine $\wp(x_4)$, giving an Artin-Schreier equation
- Redefine maps and elements to make definitions nicer and more consistent

Specific Items

A few definitions and identities helped this process along:

Remark

When $Gal\left(L/K\right) = \langle\sigma\rangle$, $Tr\left(L/K\right) = \sum_{i=0}^{n-1}\sigma^i$, where $n = [L : K]$

Definitions (Witt Polynomials)

$$w(x) = \frac{x^p + \wp(x)^p - (x + \wp(x))^p}{p}$$

$$W(x, y) = \frac{x^{p^2} + \wp(x)^{p^2} - (x + \wp(x))^{p^2} + p(y^p + \wp(y)^p - (y + \wp(y) + w(x))^p)}{p^2}$$

Specific Items (continued)

When $[K(x):K]=p$ and $Gal\left({}^{K(x)}/_{K}\right)=\langle\sigma\rangle$, we have:

- $Tr\left({}^{K(x)}/_{K}\right)\left(\frac{x^p+1-(x+1)^p}{p}\right)=1$
- $\wp\left(\frac{x^p+1-(x+1)^p}{p}\right)=(\sigma-1)w(x)$

When $K(x)/K$ is as above, $[K(x,y):K(x)]=p$, and $Gal\left({}^{K(x,y)}/_{K(x)}\right)=\langle\sigma^p\rangle$, we have:

- $Tr\left({}^{K(x,y)}/_{K}\right)\left(\frac{x^{p^2}+1-(x+1)^{p^2}+p\left(y^p-\left(y+\frac{x^p+1-(x+1)^p}{p}\right)^p\right)}{p^2}\right)=1$
- $\wp\left(\frac{x^{p^2}+1-(x+1)^{p^2}+p\left(y^p-\left(y+\frac{x^p+1-(x+1)^p}{p}\right)^p\right)}{p^2}\right)=(\sigma-1)W(x,y)$

Important Item

The most important item used was the additive version of Hilbert's Theorem 90, a direct result of the Normal Basis Theorem.

Hilbert's Theorem 90 (additive)

For a finite Galois extension $^L/_K$, with $Gal\left(^L/_K\right) = \langle\sigma\rangle$:

If $Tr\left(^L/_K\right)(k_1) = 0$, where $k_1 \in L$, then:
$\exists\ k_2 \in L$ such that $(\sigma - 1)k_2 = k_1$.

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Element Definitions

To simplify the results, define the following:

- $\wp(x_1) = \beta_1$, where $\beta_1 \in K$
- $\wp(x_2) = \beta_2$, where $\beta_2 \in K$
- $\wp(x_3) = B_3 + \beta_3$, where $\beta_3 \in K$, $B_3 \in K_2$
- $\wp(x_4) = B_4 + \beta_4$, where $\beta_4 \in K$, $B_4 \in K_3$
- $A_1 = \beta_2 x_1$, $A_3 = \beta_3 x_1$
- $A_2 = \beta_2 \binom{x_1}{2} = \beta_2 \frac{x_1(x_1-1)}{2}$
- $\sigma_i \in Gal\left({}^{K_4}/{}_{K}\right)$ such that σ_i fixes K_{i-1}, $\sigma_i(x_i) = x_i + 1$

Results

Extension	σ_1	σ_2	σ_3	σ_4	B_3	B_4
(vi)	d	c	b	a	0	A_1
(vii)	a	c	b	a^p	0	$-A_1 + w(x_1)$
(viii)	a	c	b	a^p	0	$A_3 + w(x_2)$
(ix), $p = 3$	c	a	b	a^p	A_1	$-A_2 - A_3 + w(x_2)$
(ix), $p > 3$	c	a	b	d	A_1	$A_2 + A_3$
(x)	a	c	b	a^p	$-A_1$	$w(x_1)$
(xi)	a	c	b	a^p	$-A_1$	$A_2 - A_3 + w(x_1)$
(xii)	a	c	b	a^p	$-A_1$	$A_2 - A_3 + w(x_1) + w(x_2)$
(xiii)	a	c	b	a^p	$-A_1$	$A_2 - A_3 + w(x_1) + nw(x_2)$
(xiv)	a	b	a^p	b^p	$-A_1 + w(x_1)$	$w(x_2)$
(xv)	a	b	a^p	a^{p^2}	$w(x_1)$	$-A_1 + W(x_1, x_3)$